Assignment 9.

1. (a) By sketching a suitable pair of graphs, show that there is only one value of x that is a root of the following equation [2]

$$\frac{x}{3} + 2 = e^{-x}.$$

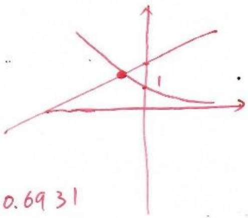
(b) Verify, by calculation, that this root lies between -1 and 0. [2]

(c) Show that, if a sequence of values given by the iterative formula

$$x_{n+1} = \ln 3 - \ln(x_n + 6)$$

converges, then it converges to the root of the equation given in part (a). [2]

(d) Use this iterative formula, with $x_1 = 0$, to calculate the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]



(b) $f(x) = \frac{x}{3} + 2 - e^{-x}$

$f(-1) = -1.05 < 0$

$f(0) = 1 > 0$

So there is ~~x~~, $f(x) = 0$. $x \in (-1, 0)$

this root lies between -1 and 0.

(d)

$x_1 = 0$

$x_2 = -0.6931$

$x_3 = -0.5704$

$x_4 = -0.5933$

$x_5 = -0.5890$

$x_6 = -0.5898$

$\Rightarrow \alpha = -0.59$

(c) $x_n \to \alpha, \Rightarrow x_{n+1} \to \alpha$

hence $\alpha = \ln 3 - \ln(\alpha + 6)$

$e^{\alpha} = \frac{3}{\alpha + 6} \Rightarrow e^{-\alpha} = \frac{\alpha + 6}{3}$

$\Rightarrow \frac{\alpha}{3} + 2 = e^{-\alpha}$

2. The constant a, where $a > 1$, is such that $\int_1^a \left(\sqrt{x} + \frac{1}{x}\right) dx = 10.$

(a) Find an equation satisfied by a, and show that it can be written in the form [5]

$$a = \left(16 - \tfrac{3}{2}\ln a\right)^{\frac{2}{3}}.$$

(b) Verify, by calculation, that the equation $a = \left(16 - \frac{3}{2}\ln a\right)^{\frac{2}{3}}$ has a root between 5 and 6. [2]

(c) Use the iterative formula

$$a_{n+1} = \left(16 - \tfrac{3}{2}\ln a_n\right)^{\frac{2}{3}},$$

with $a_1 = 5$, to calculate the value of a correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

(a) $\left[\frac{2}{3}x^{\frac{3}{2}} + \ln|x|\right]_1^a = \frac{2}{3}a^{\frac{3}{2}} + \ln a - \frac{2}{3} = 10$

$\Rightarrow a = \left(16 - \frac{3}{2}\ln a\right)^{\frac{2}{3}}$

(b) Let $f(a) = a - \left(16 - \frac{3}{2}\ln a\right)^{\frac{2}{3}}$

$f(5) = -0.69 < 0$, $f(6) = 0.38 > 0$

$\Rightarrow a = 5.64$

So $f(a) = 0$ has a root between 5 and 6.

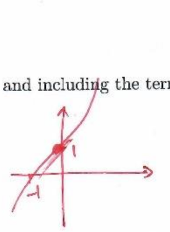
that is $a = \left(16 - \frac{3}{2}\ln a\right)^{\frac{2}{3}}$ has root between 5 and 6.

(c) $a_1 = 5$, $a_2 = 5.6937$, $a_3 = 5.6341$, $a_4 = 5.6431$, $a_5 = 5.6478$,

3. A curve has equation $f(x) = \sqrt{1+2x^3}$, for $-0.5 \le x \le 0.5$.

(a) Use the binomial expansion to express $f(x)$ in ascending powers of x, up to and including the term in x^6. Hence prove that $f(x) \approx 1 + x^3$ when $|x|$ is sufficiently small. [3]

(b) Use the trapezium rule with four intervals to estimate the value of



$$\int_{-0.5}^{0.5} \sqrt{1+2x^3}\,\mathrm{d}x,$$

giving your answer correct to 2 decimal places. [3]

(c) Sketch the graph of $y = 1 + x^3$, labeling the points where the curve meets the x- and y-axes. [2]

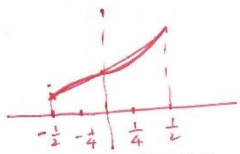
(d) Now assume that the graph of $f(x)$ is similar to the graph of $y = 1 + x^3$, for $-0.5 \le x \le 0.5$. Explain, with reference to your graph in part (c), why the estimate in part (b) may be expected to give a good approximation to the true value of the integral in this case. [1]

(a) $(1+2x^3)^{\frac12} = 1 + \frac12 \cdot 2x^3 + \frac{\frac12(-\frac12)}{2!}(2x^3)^2$

$= 1 + x^3 - \frac18 \cdot 4x^6 + \cdots$

$= 1 + x^3 - \frac12 x^6$

(b)



$\frac14\left(\frac12 f(-0.5) + \frac12 f(0.5) + f(-0.25) + f(0) + f(0.25)\right)$

$= 0.9979 \approx 1.00$

(d) $f''(x) < 0$ for $-0.5 \le x < \cancel{0.5}\ 0$

$f'(x) > 0$. $0 < x < -0.5$

4. A curve has equation $y = \sqrt{2x+1}$, for $x \ge -\frac12$.

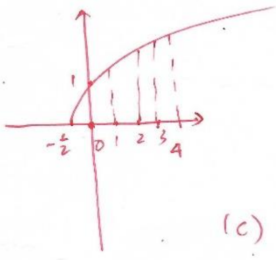
(a) Sketch the curve, labeling the points where the curve meets the x- and y-axes. [2]

(b) Evaluate the area under the curve between the lines $x = 0$ and $x = 4$. [3]

(c) Use the trapezium rule with four intervals to estimate the area in part (b), giving your answer correct to 2 decimal places. [3]

(d) The estimate found in part (c) is denoted by E. Explain, without further calculation, whether another estimate found using the trapezium rule with eight intervals would be greater than E or less than E. [2]

(a)



$\int_0^4 \sqrt{2x+1}\,dx$

$= \left[\frac23 \cdot \frac12 (2x+1)^{\frac32}\right]_0^4 = \frac13 \cdot 27 - \frac13 \cdot 1 = \frac{26}{3}$

(c) $\left(\frac12\left(f(0)+f(4)\right) + f(1) + f(2) + f(3)\right)$

$= 0.5 \times (1+3) + \sqrt3 + \sqrt5 + \sqrt7 = 8.61$

(d) greater.

Total mark of this assignment: 38.